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HYERS-ULAM STABILITY OF THE QUADRATIC AND JENSEN FUNCTIONAL EQUATIONS ON UNBOUNDED DOMAINS

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Abstract

In the present paper, we investigate the Hyers-Ulam stability of the quadratic functional equation and the Jensen functional equation on unbounded domains.

1. Introduction

In 1940, Ulam [30] gave a wide ranging talk before the mathematics club of the University of Wisconsin, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

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Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that, if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all x, y in G_1, then there exists a homomorphism $a : G_1 \to G_2$ such that $d(a(x), f(x)) \leq \epsilon$ for all x in G_1?

The case of approximately additive functions was solved by Hyers [8] under the condition that G_1 and G_2 are Banach spaces. Taking this fact into account, the Cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to have *Hyers-Ulam stability* on (G_1, G_2). The result of Hyers was significantly generalized by Rassias [19]. Since then, the stability of several functional equations have been investigated. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds.

It should be remarked that, we can find in [5], [11], [24], and [25] a lot of references concerning the Hyers-Ulam-Rassias stability of functional equations, (see also [3], [4], [6], [7], [9], [14], [20], [21], and [23]).

A stability problem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x, y \in E, \tag{1.1}$$

was proved by Skof [26], and later by Jung [13] on unbounded domains.

Equation (1.1) has been generalized by Stetkaer [29] to the more general equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in E, \tag{1.2}$$

where $\sigma : E \to E$ is an automorphism of the normed space E such that $\sigma \circ \sigma = I$, (I denotes the identity).

Recently, the stability theorem of Equation (1.2) and the Jensen functional equations

$$f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in E, \tag{1.3}$$

$$f(x + y) - f(x + \sigma(y)) = 2f(y), \quad x, y \in E, \tag{1.4}$$

has been proved, (see [2], [16], and [17]).

In [18], the authors investigated the stability of Equations (1.2) and (1.3) on unbounded domains: $\{(x, y) \in E^2 : \|y\| \geq d\}$.

The stability problems of several functional equations on a restricted domains have been extensively investigated by a number of authors, we refer, for example, to [9], [12], [15], [27], and [28].

Our main goal in this paper is to investigate the Hyers-Ulam stability problem for the Equations (1.2), (1.3), and (1.4) on unbounded domains: $\{(x, y) \in E^2 : \|x\| \geq d\}$.

2. Hyers-Ulam Stability of Equation (1.2) on Unbounded Domains

In this section, we will investigate the Hyers-Ulam stability of Equation (1.2) on unbounded domains: $\{(x, y) \in E^2 : \|x\| \geq d\}$.

Theorem 2.1. *Let* $d \geq 0$, $\delta \geq 0$, *and* $\gamma \geq 0$ *be given. Assume that a mapping* $f : E \to F$ *satisfies the inequalities*

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)\| \leq \delta, \tag{2.1}$$

$$\|f(x) - f(\sigma(x))\| \leq \gamma, \tag{2.2}$$

for all $x, y \in E$ *with* $\|x\| \geq d$. *Then, there exists a unique solution* $q : E \to F$ *of Equation* (1.2) *such that*

$$\|f(x) - q(x)\| \leq \frac{5\delta}{2} + \frac{5\gamma}{2}, \tag{2.3}$$

for all $x \in E$.

Proof. First, we will prove that the function $x \mapsto \|f(x) - f(\sigma(x))\|$ is bounded on E. Let $x \in E$ such that $\|x\| < d$ and $\sigma(x) \neq x$, and let $z = 2^n x$ with n large enough, so $\|z\| > d$, $\|\sigma(z)\| > d$, and $\|x - z\| > d$. By using now, the inequalities (2.1), (2.2), the triangle inequality, and the following equation

$$\begin{aligned} f(x) - f(\sigma(x)) = & -[f(\sigma(z) + x - z) + f(\sigma(x)) - 2f(\sigma(z)) - 2f(x - z)] \\ & +[f(z + \sigma(x) - \sigma(z)) + f(x) - 2f(z) - 2f(\sigma(x) - \sigma(z))] \end{aligned}$$

$$+2[f(z) - f(\sigma(z))] + [f(\sigma(z) + x - z) - f(z + \sigma(x) - \sigma(z))]$$

$$+2[f(\sigma(x) - \sigma(z)) - f(x - z)],$$

we get

$$\|f(x) - f(\sigma(x)\| \leq 2\delta + 5\gamma, \tag{2.4}$$

for all $x \in E$. Now, we will show that the function: $(x, y) \mapsto \|f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)\|$ is bounded on E. Let $x, y \in E$ such that $\|x\| < d$. If $x = 0$, then we have $\|f(0 + y) + f(0 + \sigma(y)) - 2f(0) - 2f(y)\| = \|f(\sigma(y)) - f(y) - 2f(0)\| \leq 2\delta + 5\gamma + 2\|f(0)\|$. For $x \neq 0$, we choose $z = 2^n x$, $n \in \mathbb{N}$, and we discuss the following cases.

Case 1. $\sigma(x) \neq -x$.

With n large enough, we have $\|z\| > d, \|x + z\| > d, \|\sigma(x) + z\| > d$, $\|y + \sigma(z)\| > d$, and $\|z + \sigma(z)\| > d$. By using (2.1), (2.2), the triangle inequality, and the following decomposition:

$$\begin{aligned}
&2[f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)] \\
&= [f(x + z + y + \sigma(z)) + f(x + z + \sigma(y) + z) - 2f(x + z) - 2f(y + z)] \\
&\quad -[f(2z + x + \sigma(y)) + f(2z + \sigma(x) + y) - 2f(2z) - 2f(x + \sigma(y))] \\
&\quad +[f(\sigma(x) + z + y + z) + f(\sigma(x) + z + \sigma(y) + \sigma(z)) - 2f(\sigma(x) + z) - 2f(y + z)] \\
&\quad +2[f(z + x) + f(z + \sigma(x)) - 2f(z) - 2f(x)] \\
&\quad +2[f(z + y) + f(z + \sigma(y)) - 2f(z) - 2f(y)] \\
&\quad -2[f(2z) + f(z + \sigma(z)) - 2f(z) - 2f(z)] + 2[f(y + \sigma(z)) - f(z + \sigma(y))] \\
&\quad -[f(z + \sigma(z) + x + y) + f(z + \sigma(z) + \sigma(x) + \sigma(y)) - 2f(z + \sigma(z)) - 2f(x + y)],
\end{aligned}$$

we obtain

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)\| \leq 5\delta + \gamma. \tag{2.5}$$

Case 2. $\sigma(x) = -x$.

In this case, we use the following relation:

$$\begin{aligned}
&2[f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)] \\
&\quad = -[f(2z+x+y)+f(2z-x+\sigma(y))-2f(2z)-2f(x+y)] \\
&\qquad -[f(2z+x+\sigma(y))+f(2z-x+y)-2f(2z)-2f(x+\sigma(y))] \\
&\qquad +[f(-x+2z+y)+f(-x+2z+\sigma(y))-2f(-x+2z)-2f(y)] \\
&\qquad +[f(x+2z+y)+f(x+2z+\sigma(y))-2f(x+2z)-2f(y)] \\
&\qquad +2[f(2z+x)+f(2z-x)-2f(2z)-2f(x)],
\end{aligned}$$

and we get

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \le 3\delta. \tag{2.6}$$

If $x = y = 0$, then we choose an arbitrary $z \in E$ with $\|z\| = d$. So, by using the above decomposition (Case 2), we get $\|2f(0)\| \le 3\delta$. Consequently, the inequality

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \le 5\delta + 5\gamma,$$

holds for all $x, y \in E$. Now, in view of [1], we get the rest of the proof. □

Corollary 2.2. *A mapping* $f : E \to F$ *is a solution of Equation* (1.2), *if and only if*

$$\|f(x)-f(\sigma(x))\| \to 0 \ \textit{and} \ \sup_{y\in E}\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \to 0, \tag{2.7}$$

as $\|x\| \to +\infty$.

Proof. According to our asymptotic condition, there exist two sequences (δ_n) and γ_n, monotonically decreasing to zero such that

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \le \delta_n,$$

$$\|f(x)-f(\sigma(x))\| \le \gamma_n,$$

for all $x, y \in E$ with $\|x\| \geq n$. By Theorem 2.1, it follows that there exists a unique solution of Equation (1.2): $q_n : E \to F$ such that

$$\|f(x) - q_n\| \leq \frac{5\delta_n}{2} + \frac{5\gamma_n}{2}, \tag{2.8}$$

for all $x \in E$. Let n and m be integers satisfying $m > n > 0$. In view of (2.8), we get

$$\|f(x) - q_m\| \leq \frac{5\delta_m}{2} + \frac{5\gamma_m}{2} \leq \frac{5\delta_n}{2} + \frac{5\gamma_n}{2}, \tag{2.9}$$

for all $x \in E$. Consequently, by using the uniqueness of q_n, we obtain $q_m = q_n$ for all $n, m \in \mathbb{N}$. Finally, by letting $n \to +\infty$, we get that f is a solution of Equation (1.2). The reverse assertion is obvious. □

Corollary 2.3 $(\sigma = -I)$. *Let* $d \geq 0, \gamma \geq 0,$ *and* $\delta \geq 0$ *be given. Assume that a mapping* $f : E \to F$ *satisfies the inequalities*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta, \tag{2.10}$$

$$\|f(x) - f(-x)\| \leq \gamma, \tag{2.11}$$

for all $x, y \in E$ *with* $\|x\| \geq d$. *Then, there exists a unique solution* $q : E \to F$ *of the quadratic functional equation* (1.1) *such that*

$$\|f(x) - q(x)\| \leq \frac{5\delta}{2} + \frac{5\gamma}{2}, \tag{2.12}$$

for all $x \in E$.

Corollary 2.4. *A mapping* $f : E \to F$ *is a solution of* (1.1), *if and only if*

$$\|f(x) - f(-x)\| \to 0 \text{ and } \sup_{y \in E}\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \to 0, \tag{2.13}$$

as $\|x\| \to +\infty$.

Corollary 2.5 $(\sigma = I)$. *Let* $d > 0$ *and* $\delta > 0$ *be given. Assume that a mapping* $f : E \to F$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \tag{2.14}$$

for all $x, y \in E$ *with* $\|x\| \geq d$. *Then, there exists a unique additive mapping* $A : E \to F$ *such that*

$$\|f(x) - A(x)\| \leq 5\delta, \tag{2.15}$$

for all $x \in E$.

Corollary 2.6. *A mapping* $f : E \to F$ *is additive, if and only if*

$$\sup_{y \in E} \|f(x+y) - f(x) - f(y)\| \to 0, \tag{2.16}$$

as $\|x\| \to +\infty$.

By using the proof of Theorem 2.1 and Corollary 2.2, we get the following results.

Corollary 2.7 [18]. *A mapping* $f : E \to F$ *is a solution of Equation* (1.2), *if and only if*

$$\sup_{x \in E} \|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \to 0, \tag{2.17}$$

as $\|y\| \to +\infty$.

Corollary 2.8 [18]. *A mapping* $f : E \to F$ *is a solution of Equation* (1.2), *if and only if*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \to 0, \tag{2.18}$$

as $\|x\| + \|y\| \to +\infty$.

Corollary 2.9 [13]. *A mapping* $f : E \to F$ *is a solution of Equation* (1.1), *if and only if*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \to 0, \tag{2.19}$$

as $\|x\| + \|y\| \to +\infty$.

Corollary 2.10 [28]. *A mapping* $f : E \to F$ *is additive, if and only if*

$$\|f(x+y) - f(x) - f(y)\| \to 0, \tag{2.20}$$

as $\|x\| + \|y\| \to +\infty$.

3. Hyers-Ulam Stability of Equation (1.3) on Unbounded Domains

In this section, we establish the Hyers-Ulam stability theorem for Equation (1.3) on unbounded domains: $\{(x, y) \in E^2 : \|x\| \geq d\}$.

Theorem 3.1. *Let* $d \geq 0$, $\delta \geq 0$, *and* $\gamma \geq 0$ *be given. Assume that a mapping* $f : E \to F$ *satisfies the inequalities*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \leq \delta, \tag{3.1}$$

$$\|f(x) + f(\sigma(x))\| \leq \gamma, \tag{3.2}$$

for all $x, y \in E$ *with* $\|x\| \geq d$. *Then, there exists a unique additive mapping* $J : E \to F$ *as a solution of Equation* (1.3) *such that* $J(\sigma(x)) = -J(x)$, *and*

$$\|f(x) - f(0) - J(x)\| \leq 12\delta + 9\gamma, \tag{3.3}$$

for all $x \in E$.

Proof. Let us show that the function $x \mapsto \|f(x) + f(\sigma(x))\|$ is bounded on E. For each $x \in E$, such that $0 < \|x\| < d$ and for $z = 2^n x$ with n large enough, we have.

Case 1. $\sigma(x) \neq x$. From $\|z\| > d$, $\|\sigma(z)\| > d$, and $\|x + \sigma(z) - z\| > d$, the inequalities (3.1), (3.2), the triangle inequality, and the following equation

$$\begin{aligned} f(x) + f(\sigma(x)) = &\left[f(\sigma(z) + x - z) + f(\sigma(x)) - 2f(\sigma(z))\right] \\ &+\left[f(z + \sigma(x) - \sigma(z)) + f(z + x - z) - 2f(z)\right] \\ &+2\left[f(z) + f(\sigma(z))\right] - \left[f(\sigma(z) + x - z) + f(z + \sigma(x) - \sigma(z))\right], \end{aligned}$$

it follows that

$$\|f(x) + f(\sigma(x)\| \leq 2\delta + 3\gamma. \tag{3.4}$$

Case 2. $\sigma(x) = x$. From the following relation,

$$f(x) + f(\sigma(x)) = [f(z + x - z) + f(z + \sigma(x) - (z)) - 2f(z)]$$
$$+[f(z) + f(\sigma(z))],$$

we get

$$\|f(x) + f(\sigma(x)\| \leq \delta + \gamma. \tag{3.5}$$

Consequently, we have

$$\|f(x) + f(\sigma(x)\| \leq 2\delta + 3\gamma, \tag{3.6}$$

for all $x \in E - \{0\}$.

Now, we prove that the function: $(x, y) \mapsto \|f(x + y) + f(x + \sigma(y)) - 2f(x)\|$ is bounded on E. Let $x, y \in E$ such that $\|x\| < d$. If $x = 0$, then by using (3.6), we obtain

$$\|f(0 + y) + f(0 + \sigma(y)) - 2f(0)\| \leq 2\delta + 3\gamma + 2\|f(0)\|. \tag{3.7}$$

For $x \neq 0$, we choose $z = 2^n x$, $n \in \mathbb{N}$, and we discuss the following cases.

Case 1. $\sigma(x) \neq x$. For n large enough, we can easily verify that $\|x - z\| \geq d$, $\|x - \sigma(z)\| \geq d$, $\|x - z + \sigma(z)\| \geq d$, and $\|x - \sigma(z) + y + z\| \geq d$. Therefore, from (3.1), (3.2), the triangle inequality, and the following relation,

$$f(x + y) + f(x + \sigma(y)) - 2f(x)$$
$$= [f(x + y) + f(x - z + \sigma(y) + \sigma(z)) - 2f(x - z)]$$
$$+[f(x - \sigma(z) + y + z) + f(x + \sigma(y)) - 2f(x - \sigma(z))]$$
$$-[f(x) + f(x - z + \sigma(z)) - 2f(x - z)]$$
$$-[f(x) + f(\sigma(x) - \sigma(z) + z) - 2f(z)]$$
$$-[f(x - z + \sigma(y) + \sigma(z)) + f(\sigma(x) + y) - 2f(\sigma(z))]$$

$$+[f(\sigma(x)+y)+f(\sigma(x)-z+\sigma(y)+\sigma(z))-2f(\sigma(x)-z)]$$

$$+2[f(x-\sigma(z))+f(\sigma(x)-z)]-2[f(z)+f(\sigma(z))]$$

$$+[f(x-z+\sigma(z))+f(\sigma(x)-\sigma(z)+z)]$$

$$-[f(x-\sigma(z)+y+z)+f(\sigma(x)-z+\sigma(y)+\sigma(z))],$$

we get

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\| \le 6\delta+6\gamma. \tag{3.8}$$

Case 2. $\sigma(x)=x$. By using (3.1), (3.2), and the following decomposition

$$f(x+y)+f(x+\sigma(y))-2f(x)$$

$$=[f(x-z+y+z)+f(x-z+\sigma(y)+z)-2f(x-z)]$$

$$-[f(-z+x+z)+f(-z+\sigma(x)+z)-2f(-z)]$$

$$+[f(-z+x)+f(-z+x)-2f(-z)],$$

we get

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\| \le 3\delta. \tag{3.9}$$

If $x=0=y$, then we choose an arbitrary $z\in E$ such that $\|z\|=d$. By using the above decomposition (Case 2), we obtain $\|f(0)\|\le 3\delta$.

Finally, the inequality

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\| \le 8\delta+6\gamma, \tag{3.10}$$

holds true for all $x, y\in E$. Now, from [16], one gets that there exists a unique additive mapping $J: E\to F$, which satisfies the inequality (3.3). Furthermore, $J(\sigma(x))=-J(x)$ for all $x\in E$. This completes the proof of theorem. □

Corollary 3.2. *A mapping* $f: E\to F$ *is a solution of Equation* (1.3), *if and only if*

$$\|f(x) + f(\sigma(x))\| \to 0 \text{ and } \sup_{y \in E}\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \to 0, \quad (3.11)$$

as $\|x\| \to +\infty$.

Corollary 3.3 $(\sigma = -I)$**.** *Let* $d > 0, \gamma \geq 0,$ *and* $\delta > 0$ *be given. Assume that a mapping* $f : E \to F$ *satisfies the inequalities*

$$\|f(x + y) + f(x - y) - 2f(x)\| \leq \delta, \quad (3.12)$$

$$\|f(x) + f(-x)\| \leq \gamma, \quad (3.13)$$

for all $x, y \in E$ with $\|x\| \geq d$. *Then, there exists a unique additive mapping* $J : E \to F$ *solution of the Jensen functional equation* (1.3) *such that* $J(-x) = -J(x)$, *and*

$$\|f(x) - f(0) - J(x)\| \leq 12\delta + 9\gamma, \quad (3.14)$$

for all $x \in E$.

Corollary 3.4. *A mapping* $f : E \to F$ *is a solution of* (1.3), *if and only if*

$$\|f(x) + f(-x)\| \to 0 \text{ and } \sup_{y \in E}\|f(x + y) + f(x - y) - 2f(x)\| \to 0, \quad (3.15)$$

as $\|x\| \to +\infty$.

Corollary 3.5 [18]**.** *A mapping* $f : E \to F$ *is a solution of* (1.3), *if and only if*

$$\sup_{x \in E}\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \to 0, \quad (3.16)$$

as $\|y\| \to +\infty$.

Corollary 3.6 [18]**.** *A mapping* $f : E \to F$ *is a solution of* (1.3), *if and only if*

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \to 0, \quad (3.17)$$

as $\|x\| + \|y\| \to +\infty$.

4. Hyers-Ulam Stability of Equation (1.4) on Unbounded Domains

In this section, we investigate the stability of the Jensen functional equation (1.4) on a restricted domain: $\{(x, y) \in E^2 : \|x\| \geq d\}$.

Theorem 4.1. *Let $d \geq 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y) - f(x+\sigma(y)) - 2f(y)\| \leq \delta, \tag{4.1}$$

for all $x, y \in E$ with $\|x\| \geq d$. Then, there exists a unique additive mapping $j : E \to F$ solution of the Jensen functional equation (1.4) *such that $j(\sigma(x)) = -j(x)$, and*

$$\|f(x) - j(x)\| \leq 3\delta, \tag{4.2}$$

for all $x \in E$.

Proof. Let $x, y \in E$ such that $0 < \|x\| < d$. We choose $z = 2^n x$, where n is large enough, so $\|z\| \geq d$, $\|x + z\| \geq d$, and $\|z + \sigma(x)\| \geq d$. From (4.1), the triangle inequality, and the following decomposition,

$$\begin{aligned}
&2[f(x+y) - f(x+\sigma(y)) - 2f(y)] \\
&\quad = -[f(z+x+y) - f(z+\sigma(x)+\sigma(y)) - 2f(x+y)] \\
&\qquad +[f(z+x+\sigma(y)) - f(z+\sigma(x)+y) - 2f(x+\sigma(y))] \\
&\qquad +[f(x+z+y) - f(x+z+\sigma(y)) - 2f(y)] \\
&\qquad +[f(z+\sigma(x)+y) - f(z+\sigma(x)+\sigma(y)) - 2f(y)],
\end{aligned}$$

we have

$$\|f(x+y) - f(x+\sigma(y)) - 2f(y)\| \leq 2\delta, \tag{4.3}$$

for all $x, y \in E$ with $x \neq 0$. Now, if $x = 0$, the following relation with an arbitrary $z \in E$ such that $\|z\| = d$

$$
\begin{aligned}
&2[f(0+y)-f(0+\sigma(y))-2f(y)]\\
&\quad=-[f(z+x+y)-f(z+\sigma(y))-2f(y)]\\
&\quad\quad+[f(z+\sigma(y))-f(z+y)-2f(\sigma(y))]\\
&\quad\quad+2[f(z+y)-f(z+\sigma(y))-2f(y)],
\end{aligned}
$$

implies that

$$\|f(0+y)-f(0+\sigma(y))-2f(y)\| \leq 2\delta,$$

for all $y \in E$. Consequently,

$$\|f(x+y)-f(x+\sigma(y))-2f(y)\| \leq 2\delta,$$

for all $x, y \in E$, so from [17], there exists a unique additive mapping $j : E \to F$ such that $j(\sigma(x)) = -j(x)$ and $\|f(x)-j(x)\| \leq 3\delta$. This ends our proof. □

Corollary 4.2. *A mapping $f : E \to F$ is a solution of Equation* (1.4), *if and only if*

$$\sup_{y\in E}\|f(x+y)-f(x+\sigma(y))-2f(y)\| \to 0, \tag{4.4}$$

as $\|x\| \to +\infty$.

Corollary 4.3 $(\sigma = -I)$. *Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y)-f(x-y)-2f(y)\| \leq \delta, \tag{4.5}$$

for all $x, y \in E$ with $\|x\| \geq d$. Then, there exists a unique additive mapping $j : E \to F$ solution of the Jensen functional equation (1.4) *such that $j(-x) = j(x)$ and*

$$\|f(x)-j(x)\| \leq 3\delta, \tag{4.6}$$

for all $x \in E$.

Corollary 4.4. *A mapping $f : E \to F$ is a solution of* (1.4), *if and only if*

$$\sup_{y\in E}\|f(x+y)-f(x-y)-2f(y)\| \to 0, \tag{4.7}$$

as $\|x\| \to +\infty$.

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